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NOTE

A Short Note on Some Tractable Cases of the Satisfiability Problem

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It is shown that the tractable class of CNF formulas solvable by linear autarkies properly contains the class of q-Horn formulas and that it is incomparable with SLUR.

Key Words: satisfiability; autarkies; horn; complexity.

1. INTRODUCTION

The reader is supposed to be familiar with the basic terminology and facts regarding the satisfiability problem (see [7] for a survey).

The two most well-known tractable classes of satisfiability problems are the Horn formulas and the 2-Sat formulas, which are in fact solvable in linear time; see [6] and [1], respectively. Several tractable extensions of the class of Horn formulas are studied in the literature, such as renamable Horn [10], extended Horn [4], "balanced" [5, 14], and SLUR [12]. The latter abbreviation stands for single lookahead unit resolution. It is shown in [12] that SLUR contains Horn, renamable Horn, extended Horn, and balanced. In fact, SLUR is defined by the successful output of an incomplete algorithm, also called SLUR. This algorithm is solely based on a sequence of unit propagations, without performing backtrackings. In doing so, they are able to avoid the associated recognition problems for the various Horn-like extensions mentioned.

Another extension is q-Horn, introduced in [3, 13]. This class is defined by the outcome of a certain linear program, as will be discussed later, and contains the Horn- and 2-Sat formulas. It is shown in [3] that q-Horn formulas are solvable in polynomial time: the outcome of the linear program gives rise to a certain partitioning of the set of variables and, given this partitioning, the formula can be solved for satisfiability in linear time.

In [2] this same linear program is used to define the complexity index of a CNF formula. This index sharply delineates between "easy" and "hard" problems: it is

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shown that the class of problems with a complexity index smaller than $1=\varepsilon$ ($\varepsilon>0$ fixed) still constitutes an NP-complete class.

Franco [8] gives an alternative characterization of q-Horn, avoiding the aspects from linear programming, and he shows that SLUR and q-Horn are incomparable extensions of Horn. Finally, Kullmann [9] introduces the class of formulas solvable through linear autarkies (LinAut). In short, a linear autarky for a CNF formula is the outcome of an associated linear program (related to that of [3] but not the same) which defines an autark partial assignment by a rounding procedure. It is shown in [9] that a CNF formula can be reduced to a satisfiable equivalent and linear autarky free subformula, in polynomial time.

In this note we prove that LinAut properly contains q-Horn but is incomparable with SLUR. As corollaries to our main lemma we also include a few results on the complexity index [2].

2. Q-HORN FORMULAS

We shall use the characterization of q-Horn formulas by means of the complexity index of [2]. However, we choose our formulations in such a way that they apply to Boolean values -1 and $+1$ of the propositional variables, rather than the more customary 0 and 1. The concept of linear autarky is easier to understand in this way, because the description of the underlying linear algebra is more transparent in doing so.

The *clause-variable matrix* A associated with a CNF formula Φ is the matrix defined by

$$A_{i,j}=\begin{cases} 1 & \text{if the } i\text{th clause of } \Phi \text{ contains the } j\text{th variable with sign 1 (nonnegated)} \\ -1 & \text{if the } i\text{th clause of } \Phi \text{ contains the } j\text{th variable with sign } -1 \text{ (negates)} \\ 0 & \text{otherwise.} \end{cases}$$

Thus if Φ is a CNF formula with propositional variables $x_1, ..., x_n$ the satisfiability problem for Φ reads as the $-1, 1$ feasibility problem

$$\begin{cases} Ax \geqslant -L+2e \\ x\in\{-1,1\}^n. \end{cases}$$

In the above, L is the *length vector*, having the length of clause i as its ith entry, and e is the all-one vector of appropriate dimension.

The *complexity index* of a formula with clause-variable matrix A is the (optimal) solution Z of the *linear program*

$$\begin{cases} \min Z \\ Ax \geqslant L-2Ze \\ x\in[-1,1]^n \end{cases}$$

(after substituting $\alpha_j = \frac{1}{2}(1 - x_j)$ in the definition of [2]). Note that $Z \geqslant 0$, since no feasible solutions in x and Z exist with $Z < 0$. Hence the above linear program is bounded from below and a solution exists. Alternatively, Z is the minimum over all n-dimensional vectors x with entries in the interval $[-1, 1]$ of the maximal entry of the vector $\frac{1}{2}(L - Ax)$. Also note that the complexity index cannot decrease by adding clauses to a CNF formula.

A formula is q-Horn if its complexity index $Z \leqslant 1$.

3. LINEAR AUTARKIES

First, we recall that an autarky [11] for a CNF formula ϕ is a partial assignment which satisfies all those clauses of ϕ affected by it. If one deletes all clauses affected by an autarky, a satisfiability equivalent formula is obtained. That is, the resulting formula is satisfiable if and only if the original is such.

After Kullmann [9] a formula with clause-variable matrix A has a linear autarky $x \in Q^n$ if

$$\begin{cases} Ax \geqslant 0 \\ x \neq 0. \end{cases}$$

The above concept generalizes an earlier version of Warners and van Maaren [15] which provides a decomposition of a formula in case the kernel of its clause-variable matrix is nonzero.

A so-called *monotone variable* is the most well-known example of a linear autarky: If variable x_j appears only positively (negatively) in the formula, the vector $x = e_j$ $(x = -e_j)$, (e_j is the jth unit vector) is a linear autarky.

In general, a linear autarky x leads to an autark partial assignment of the formula involved by rounding. To see this, let the partial assignment **sign** be

$$\mathbf{sign}(x_j) = \begin{cases} 1 & \text{if} \quad x_j > 0 \\ -1 & \text{if} \quad x_j < 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

and substitute all defined $\mathbf{sign}(x_j)$ into the formula. If a clause i is affected by this substitution, that is, if a variable with defined **sign** occurs in clause i, this clause is obviously satisfied by the partial assignment since

$$\sum_j A_{ij} x_j = \sum_{x_j \neq 0} A_{ij} x_j \geqslant 0$$

and hence not all $A_{ij} x_j$ with defined $\mathbf{sign}(x_j)$ can be negative.

The above means that a linear autarky leads to a nontrivial decomposition of the formula involved. One part is satisfied by **sign** and the other part contains only variables with undefined **sign**. Clearly, the latter part is satisfiability equivalent to the original formula. Kullmann [9] shows, among other things, the following:

(i) Linear programming guarantees a polynomial time search for linear autarkies.

(ii) By repeatedly applying the above decomposition, a formula is satisfiable equivalent to a linear autarky-free subformula. This subformula is unique and computable in polynomial time.

(iii) A (renamable) Horn formula without unit clauses is completely solved by (ii); that is, the resulting subformula is empty.

(iv) Every *satisfiable* 2-SAT formula is completely solved by (ii) (note that in this case a satisfying assignment is in fact a linear autarky). In other words, if the resulting linear autarky-free subformula is nonempty, the formula is unsatisfiable.

In the following, a formula is said to be in the class LinAut if the procedure described in (ii) above results in a complete assignment (the satisfiable case), or in a linear autarky-free 2-SAT subformula (the unsatisfiable case).

It is also assumed that CNF formulas have *no* unit clauses.

4. THE MAIN LEMMA AND ITS COROLLARIES

LEMMA. *Suppose a CNF formula has complexity index Z and no linear autarkies. Then either*

(a) *For some clause i (with length L_i) we have $Z > \frac{1}{2} L_i$, or*

(b) *All clauses have the same length and $L = 2Ze$.*

Proof. If (a) is not true we clearly have $L - 2Ze \geqslant 0$. Hence by definition of the complexity index we have $Ax \geqslant L - 2Ze \geqslant 0$ for some $x \in [-1, 1]^n$. Since no linear autarkies exist we must have $x = 0$ which yields in turn $L = 2Ze$. This proves the lemma.

COROLLARY. *A q-Horn formula without unit-clauses is in the class LinAut.*

Proof. A q-Horn formula has complexity index $Z \leqslant 1$. The unique linear autarky-free subformula of such a formula has also complexity index $Z' \leqslant 1$. Since no unit-clauses are present, case (a) of the lemma clearly not applies. Consequently, we see that the subformula is either empty, in which case the original formula has been solved completely and is satisfiable, or this subformula is in fact an unsatisfiable 2-SAT formula.

COROLLARY. *LinAut properly includes q-Horn.*

Proof. The formula with clause-variable matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

is in LinAut since it has a linear autarky $x = (1, -1, 0)$ which solves the formula at hand. It is clearly not q-Horn because

$$\begin{cases} x_1 + x_2 + x_3 \geqslant 3 - 2 \\ -x_1 - x_2 - x_3 \geqslant 3 - 2 \end{cases}$$

are contradictory.

Franco gives an example [8, Example 2.11] of a formula which is q-Horn but not SLUR. In [12] the formula with clause-variable matrix

$$B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

is used to show that SLUR properly contains extended Horn. The same example shows that SLUR and LinAut are incomparable, since $Bx \geqslant 0$ implies $x = 0$.

The following corollaries follow from the main Lemma.

Corollary. *If all clauses of a formula have length k and the formula is without linear autarkies then it has complexity index $Z = \frac{1}{2}k$.*

Proof. If all clauses have length k the complexity index Z clearly satisfies $Z \leqslant \frac{1}{2}k$ since $A0 \geqslant (k - 2 \cdot \frac{1}{2}k)\, e$. By the lemma now it follows that $Z = \frac{1}{2}k$.

Corollary. *Let k be the length of the shortest clause of a CNF formula. Then if $Z < \frac{1}{2}k$ the formula is satisfiable.*

Proof. Consider the unique linear autarky-free subformula. Since Z decreases and k increases considering subformulas, the same assumptions hold for this subformula. By the lemma, all clauses must have length $2Z$, whence $Z = \frac{1}{2}k$ according to the previous corollary. Thus the subformula at hand is empty.

5. CONCLUSIONS

This note adds to the knowledge of the hierarchy of polynomially solvable classes of satisfiability problems. It shows that LinAut properly contains q-Horn but that there is no inclusion relation with SLUR. Finally, we have shown that linear autarky-free formulas with constant clause length k have complexity index $\frac{1}{2}k$.

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REFERENCES

1. Aspvall, B., Plass, M. F., and Tarjan, R. E. (1979), A linear-time algorithm for testing the truth of certain quantified Boolean formulas, *Inform. Process. Lett.* **8**(3), 121–123.

2. Boros, E., Crama, Y., Hammer, P. L., and Saks, M. (1994), A complexity index for satisfiability problems, *SIAM J. Comput.* **23**, 45–49.

3. Boros, E., Hammer, P. L., and Sun, X. (1994), Recognition of q-Horn formulae in linear time, *Discrete Appl. Math.* **55**, 1–13.
4. Chandru, V., and Hooker, J. N. (1991), Extended Horn sets in propositional logic, *J. Assoc. Comput. Mach* **38**, 205–221.
5. Conforti, M., and Cornuéjols, G. (1995), A class of logic problems solvable by linear programming, *J. Assoc. Comput. Mach.* **42**, 1107–1113.
6. Dowling, W. F., and Gallier, J. H. (1984), Linear-time algorithms for testing the satisfiability of propositional Horn formulae, *J. Logic Programming* **1**, 267–284.
7. Du, D., Gu, J., and Pardalos, P. M. (1996), Satisfiability problem: theory and applications, Papers from the DIMACS Workshop held at Rutgers University, Piscataway, NJ.
8. Franco, J. (1997), Relative size of certain polynomial time subclasses of satisfiability, *in* "Satisfiability problem: theory and applications (Piscataway, NJ, 1996)," pp. 211–223, Amer. Math. Soc., Providence, RI.
9. Kullmann, O. Investigations on autark assignments, submitted.
10. Lewis, H. R. (1978), Renaming a set of clauses as a Horn set, *J. Assoc. Comput. Mach* **25**, 134–135.
11. Monien, B., and Speckenmeyer, E. (1985), Solving satisfiability in less than 2^n steps, *Discrete Appl. Math.* **10**, 287–295.
12. Schlipf, J. S., Annexstein, F. S., Franco, J. V., and Swaminathan, R. P. (1995), On finding solutions for extended Horn formulas, *Inform. Process. Lett.* **54**(3) 133–137.
13. Scutellà, M. G. (1990), A note on Dowling and Gallier's top-down algorithm for propositional Horn satisfiability, *J. Logic Programming* **8**, 265–273.
14. Truemper, K. (1982), Alpha-balanced graphs and matrices and gf(3)-representability of matroids, *J. Combin. Theory Ser. B* **32**, 112–139.
15. Warners, J., and van Maaren, H. (in press), Solving satisfiability problems using elliptic approximations; effective branching rules, *Discrete Appl. Math.*